

SYMMETRIC HOMOTOPY $K3$ SURFACES

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ABSTRACT. A measurement of exoticness of symplectic homotopy $K3$ surfaces is introduced, and the influence of an effective action of a $K3$ group via symplectic symmetries is investigated. In particular, it is shown that an effective action by any of the following six maximal symplectic $K3$ groups:

$$L_2(7), A_6, M_{20}, A_{4,4}, T_{192}, T_{48}$$

forces the corresponding homotopy $K3$ surface to be the least exotic.

1. INTRODUCTION

In the recent advances in topology and geometry of smooth 4-manifolds a very important role was played by one particular class of 4-manifolds, namely, the homotopy $K3$ surfaces. These manifolds have been used to test the flexibility of smooth and symplectic structures in comparison with the rigidity of holomorphic structures. To be more precise, let X be a homotopy $K3$ surface, namely, X is a closed, oriented smooth 4-manifold with $b_2^+ = 3$, which is homeomorphic to the standard $K3$ surface. If such a manifold admits an orientation-compatible symplectic structure, then it is called a symplectic homotopy $K3$ surface. While the knot surgery of Fintushel and Stern (cf. [8]) allows construction of numerous examples of symplectic homotopy $K3$ surfaces, deep work of Taubes [25] gives very strong information about the smooth structures on such manifolds. For example, one can easily show that the set of Seiberg-Witten basic classes of X spans an isotropic sublattice L_X of $H^2(X; \mathbb{Z})$ (with respect to the cup product), so that its rank, denoted by r_X , must range from 0 to 3 (cf. Theorem 4.1). The rank r_X of the lattice L_X of the Seiberg-Witten basic classes gives a rough measurement of the exoticness of the smooth structure of X , with $r_X = 0$ being the least exotic and with $r_X = 3$ being the most exotic.

There are various known characterizations of the least exotic (i.e. $r_X = 0$) symplectic homotopy $K3$ surfaces X , which are all characteristics of the standard $K3$, namely:

- X has a trivial canonical class, i.e., $c_1(K_X) = 0$, cf. [25].
- X has a unique Seiberg-Witten basic class, cf. [20, 25].
- X has the same Seiberg-Witten invariant of the standard $K3$, cf. [25].
- X is a simply-connected, minimal symplectic 4-manifold with zero Kodaira dimension, cf. [15].

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Above all, the only known example of such a 4-manifold is the standard $K3$ itself! Encouraged by this fact we shall call a symplectic homotopy $K3$ surface X with $r_X = 0$ “standard”. The main result of this paper can be considered another characterization of these manifolds as having a “large” symmetry group.

In [6] the authors have studied the possible effect of a change of a smooth structure on the symmetry group of a closed, oriented 4-manifold. It was shown that for an infinite family of the most exotic (i.e. $r_X = 3$) symplectic homotopy $K3$ surfaces, there are some significant limitations on the smooth as well as symplectic symmetry groups of these manifolds. The purpose of the current paper is to investigate the possible effect of a group action on the smooth structure of a 4-manifold.

The interaction between smooth structures and symmetry groups of a manifold is one of the basic questions in the theory of differentiable transformation groups. In particular, the following classical theorem of differential geometry gives a characterization of the standard sphere \mathbb{S}^n among all the homotopy n -spheres as having the largest degree of symmetry (cf. [12]).

Theorem (A Characterization of \mathbb{S}^n). *Let M^n be a closed, simply connected manifold of dimension n , and let G be a compact Lie group which acts smoothly and effectively on M^n . Then $\dim G \leq n(n+1)/2$, with equality if and only if M^n is diffeomorphic to \mathbb{S}^n .*

If X is a homotopy $K3$ surface then it is well known that a compact Lie group acting smoothly on X must be finite (cf. [2]). A finite group G is called a $K3$ group (resp. symplectic $K3$ group) if G can be realized as a subgroup of the automorphism group (resp. symplectic automorphism group) of a $K3$ surface. Finite automorphism groups of $K3$ surfaces were first systematically studied by Nikulin in [22]; in particular, he completely classified finite abelian groups of symplectic automorphisms. Subsequently, Mukai [21] determined all the symplectic $K3$ groups (see also [13, 26]). There are 11 maximal symplectic $K3$ groups:

$$L_2(7), A_6, S_5, M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}$$

all of which can be characterized as certain subgroups of the Mathieu group M_{23} .

Given the crucial role of homotopy $K3$ surfaces in the theory of symplectic and smooth 4-manifolds, and motivated by the above characterization of the standard \mathbb{S}^n we were led to the following

Problem *Let X be a homotopy $K3$ surface supporting an effective action of a “large” $K3$ group via symplectic symmetries. What can be said about the smooth structure on X ?*

Viewing the above maximal symplectic $K3$ groups as “large”, our solution to this problem is contained in the following:

Main Theorem *Let G be one of the following maximal symplectic $K3$ groups: $L_2(7)$, A_6 , M_{20} , $A_{4,4}$, T_{192} , T_{48} and let X be a symplectic homotopy $K3$ surface. If X*

admits an effective G -action via symplectic symmetries, then X must be “standard” (i.e., $r_X = 0$).

Remarks We would like to point out that it is possible to extend the methods employed in this paper in the following two directions:

- (1) to show that the above theorem holds for some other K3 groups which are not maximal,
- (2) to give an upper bound on the exoticness r_X when X admits a “relatively large” symplectic symmetry group.

However, we shall not pursue these extensions here as the detailed analysis depends very much on the structure of each individual group involved.

The basic idea of our methods may be summarized as follows. Let a finite group G act on a homotopy K3 surface X via symplectic symmetries. One first determines the possible fixed point set of an arbitrary element $g \in G$, from which one can compute the trace $tr(g)$ of g on $H^*(X; \mathbb{Z})$ using the Lefschetz fixed point theorem. This will then give an estimate on

$$\dim(H^*(X; \mathbb{R}))^G = \frac{1}{|G|} \sum_{g \in G} tr(g).$$

On the other hand, there is an induced action of G on the lattice L_X of the Seiberg-Witten basic classes. The following basic inequality

$$\dim(L_X \otimes_{\mathbb{Z}} \mathbb{R})^G \leq \min(b_2^+(X/G), b_2^-(X/G))$$

plus the identity $\dim(H^*(X; \mathbb{R}))^G = 2 + b_2^+(X/G) + b_2^-(X/G)$ allows one to obtain information about $\dim(L_X \otimes_{\mathbb{Z}} \mathbb{R})^G$ and $r_X = \text{rank } L_X$.

In [6] one of our results asserts that there are infinitely many symplectic homotopy K3 surfaces with $r_X = 3$ which support no symplectic symmetries by $L_2(7)$ or A_5 . The main theorem of the current paper gives a substantial improvement for the case of $L_2(7)$, however, the case of A_5 does not even follow from the methods we described above. In fact, for $G = A_5$ one can show that (cf. Lemma 2.5) $\dim(H^*(X; \mathbb{R}))^G \leq 8$. In the case of $\dim(H^*(X; \mathbb{R}))^G = 8$, one has $b_2^+(X/G) = b_2^-(X/G) = 3$. The inequality

$$r_X = \dim(L_X \otimes_{\mathbb{Z}} \mathbb{R})^G \leq \min(b_2^+(X/G), b_2^-(X/G)) = 3$$

does not yield any restriction on the exoticness r_X .

Our main theorem naturally gives rise to the following question.

What can be said about a finite group G which can act effectively on a “standard” symplectic homotopy K3 surface via symplectic symmetries?

In the following theorem, we show that the symmetries of a “standard” symplectic homotopy K3 surface look very much like holomorphic automorphisms of the standard K3 surface; in particular, the symmetry groups are more or less K3 groups.

Theorem 1.1. *Let X be a “standard” symplectic homotopy K3 surface (i.e. $r_X = 0$) and let G be a finite group acting on X via symplectic symmetries. Then there exists*

a short exact sequence of finite groups

$$1 \rightarrow G_0 \rightarrow G \rightarrow G^0 \rightarrow 1,$$

where G^0 is cyclic and G_0 is a symplectic K3 group, such that G_0 is characterized as the maximal subgroup of G with the property $b_2^+(X/G_0) = 3$. Moreover, the induced action of G_0 on X has the same fixed point set structure of a holomorphic action on the standard K3 by G_0 .

Finally, we consider K3 groups (or more generally arbitrary finite groups) which are “small” in the sense of the main theorem above. These are the finite groups which can act, via symplectic symmetries, on a symplectic homotopy K3 surface with nonzero exoticness (i.e. $r_X \geq 1$). It is fairly easy to show, using the Fintushel-Stern knot surgery [8], that if a K3 group G acts on the standard K3 such that an elliptic fibration is preserved under the action, then under a certain condition (cf. Remark 4.3) G is small in the above sense. In particular, all cyclic K3 groups of prime order can act holomorphically on an elliptic K3 surface (cf. [24, 16]), and one can show they are small by a knot surgery. Concerning noncyclic K3 groups, the following theorem perhaps gives the most dramatic example of such a construction.

Theorem 1.2. *Let $G \equiv (\mathbb{Z}_2)^3$. There exists an infinite family of distinct symplectic homotopy K3 surfaces with maximal exoticness (i.e. $r_X = 3$), such that each member of the exotic K3’s admits an effective G -action via symplectic symmetries. Moreover, the G -action is pseudofree and induces a trivial action on the lattice L_X of the Seiberg-Witten basic classes.*

In view of Theorems 1.1 and 1.2, one is left with the following two intriguing

Questions

- (1) *Are there any finite groups other than a K3 group which can act smoothly or symplectically on a homotopy K3 surface?*
- (2) *Are there any finite groups other than $(\mathbb{Z}_2)^3$ (or a subgroup of it) which can act symplectically on a homotopy K3 surface X with $r_X > 1$?*

The organization of the rest of the paper is as follows. The proof of Main Theorem and Theorem 1.1 is given in Sections 2 and 3 respectively. In Section 4 we show that the lattice L_X of Seiberg-Witten basic classes is isotropic and $r_X \leq 3$. The proof of Theorem 1.2 is also given in Section 4.

2. PROOF OF MAIN THEOREM

Let (X, ω) be a symplectic homotopy K3 surface, and let G be one of the 11 maximal symplectic K3 groups (cf. [21]), which acts on X smoothly and effectively, preserving the symplectic structure ω . We pick an arbitrary ω -compatible, G -equivariant almost complex structure J on X , and we denote by g_J the associated Riemannian metric, i.e., $g_J(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$, which is also G -equivariant.

We derive some preliminary information about the G -action first.

Lemma 2.1. *Let G_0 be the maximal subgroup of G such that $b_2^+(X/G_0) = 3$. Then G/G_0 is cyclic. In particular, the commutator $[G, G]$ is contained in G_0 .*

Proof. Let H^+ be the space of g_J -self-dual harmonic 2-forms on X . Since the Riemannian metric g_J is G -equivariant, we see that H^+ is invariant under the action of G . Moreover, since $\omega \in H^+$ and G fixes ω , we obtained an induced action of G on the orthogonal complement $\langle \omega \rangle^\perp$ of ω in H^+ . Note that $\dim H^+ = 3$, so that $\dim \langle \omega \rangle^\perp = 2$. We claim that the action of G on $\langle \omega \rangle^\perp$ is orientation-preserving (i.e. there are no reflections). The reason for this is that $b_2^+(X/G)$ must be odd, so that it is either equal to 1 or equal to 3. To see that $b_2^+(X/G)$ is odd, we note that for X/G as a symplectic 4-orbifold, the dimension of the Seiberg-Witten moduli space associated to the canonical $Spin^{\mathbb{C}}$ structure equals 0 (cf. [4], Appendix A). This gives rise to the equation

$$2 \cdot \text{index of Dirac operator} + (b_1(X/G) - 1 - b_2^+(X/G)) = 0.$$

It follows easily that $b_2^+(X/G)$ is odd because $b_1(X/G) = 0$.

With the preceding understood, we obtain an exact sequence of groups

$$1 \rightarrow G_0 \rightarrow G \rightarrow \mathbb{S}^1,$$

where the last homomorphism $G \rightarrow \mathbb{S}^1$ is induced from the action of G on $\langle \omega \rangle^\perp$. The lemma follows immediately from this. \square

The commutator $[G, G]$ and the quotient group (i.e. the abelianization) $G/[G, G]$ of a symplectic $K3$ group G is determined in [26]. The list of G where G is maximal is reproduced below.

- $G = L_2(7)$: $[G, G] = G$ and $G/[G, G] = 0$.
- $G = A_6$: $[G, G] = G$ and $G/[G, G] = 0$.
- $G = S_5$: $[G, G] = A_5$ and $G/[G, G] = \mathbb{Z}_2$.
- $G = M_{20} = 2^4 A_5$: $[G, G] = G$ and $G/[G, G] = 0$.
- $G = F_{384} = 4^2 S_4$: $[G, G] = 4^2 A_4$ and $G/[G, G] = \mathbb{Z}_2$.
- $G = A_{4,4} = 2^4 A_{3,3}$: $[G, G] = A_4^2$ and $G/[G, G] = \mathbb{Z}_2$.
- $G = T_{192} = (Q_8 * Q_8) \times_\phi S_3$: $[G, G] = (Q_8 * Q_8) \times_\phi \mathbb{Z}_3$ and $G/[G, G] = \mathbb{Z}_2$.
- $G = H_{192} = 2^4 D_{12}$: $[G, G] = 2^4 \mathbb{Z}_3$ and $G/[G, G] = (\mathbb{Z}_2)^2$.
- $G = N_{72} = 3^2 D_8$: $[G, G] = A_{3,3}$ and $G/[G, G] = (\mathbb{Z}_2)^2$.
- $G = M_9 = 3^2 Q_8$: $[G, G] = A_{3,3}$ and $G/[G, G] = (\mathbb{Z}_2)^2$.
- $G = T_{48} = Q_8 \times_\phi S_3$: $[G, G] = T_{24} = Q_8 \times_\phi \mathbb{Z}_3$ and $G/[G, G] = \mathbb{Z}_2$.

The crucial step in the proof of the main theorem is to determine the possible fixed point set of an arbitrary element of G . This is done by combining the analysis in our previous work [5] with various G -index theorems, and by exploiting the various specific features of the group G .

Here is the main technical input from [5]. Since $b_2^+(X/G_0) = 3 \geq 2$, the canonical class $c_1(K_X)$ is represented by $\sum_i n_i C_i$, where $n_i \geq 1$ and $\{C_i\}$ is a finite set of J -holomorphic curves, such that (1) $\cup_i C_i$ is invariant under the action of G_0 , (2) if $p \in X \setminus (\cup_i C_i)$ is fixed by an element $g \in G_0$, then the local representation of g at p must be contained in $SL_2(\mathbb{C})$. (In particular, p must be an isolated fixed point of g , and all the 2-dimensional components of the fixed point set $\text{Fix}(g)$ are contained in

$\cup_i C_i$.) One can further analyze the fixed point set structure of an element of G_0 by studying the induced action on $\cup_i C_i$.

Lemma 2.2. (1) *Let $g \in G$ be an involution. If $g \in G_0$, then $\text{Fix}(g)$ consists of 8 isolated fixed points. If $g \in G \setminus G_0$, then $\text{Fix}(g)$ is either empty or a disjoint union of embedded J -holomorphic curves $\{\Sigma_j\}$ such that $c_1(K_X) \cdot \Sigma_j = 0$ for each j .*

(2) *Let $g \in G_0$ be an element of order 4. Then $\text{Fix}(g)$ consists of 4 isolated fixed points, all with a local representation contained in $SL_2(\mathbb{C})$.*

Proof. (1) Since X is simply-connected, the action of g can be lifted to the spin structure, where there are two cases: (1) g is of even type, meaning that the order of the lifting is 2, and (2) g is of odd type, meaning that the order of the lifting is 4 (cf. [1]). Moreover, g has only isolated fixed points in the case of an even type, and g is free or has only 2-dimensional fixed components in the case of an odd type. On the other hand, it was shown in [3] that g is of even type, with 8 isolated fixed points, if and only if $b_2^+(X/g) = 3$, which means $g \in G_0$ in the current notation.

Now consider the case where $g \in G \setminus G_0$. In this case $\text{Fix}(g)$ is either empty or is a disjoint union of embedded surfaces Σ_j . Note that each Σ_j is J -holomorphic because we choose J to be G -equivariant.

We first show that $\sum_j c_1(K_X) \cdot \Sigma_j = 0$. To see this, suppose t is the dimension of the 1-eigenspace of g in $H^2(X; \mathbb{R})$. Then by the Lefschetz fixed point theorem and the G -signature theorem (cf. [11]), we obtain

$$\begin{cases} 2 + t - (22 - t) &= \sum_j \chi(\Sigma_j) \\ 2(2 - t) &= -16 + \sum_j \frac{2^2 - 1}{3} \cdot \Sigma_j^2, \end{cases}$$

which gives $\sum_j (\chi(\Sigma_j) + \Sigma_j^2) = 0$. By the adjunction formula, we obtain

$$\sum_j c_1(K_X) \cdot \Sigma_j = \sum_j -(\chi(\Sigma_j) + \Sigma_j^2) = 0.$$

On the other hand, $c_1(K_X) = \sum_i n_i C_i$. For any j , if $\Sigma_j \neq C_i$ for all i , then because of the positivity of intersection of J -holomorphic curves, $c_1(K_X) \cdot \Sigma_j \geq 0$ with equality iff Σ_j is disjoint from $\cup_i C_i$. If $\Sigma_j = C_i$ for some i , then $c_1(K_X) \cdot \Sigma_j = c_1(K_X) \cdot C_i = 0$ (cf. [5], Lemma 3.3). In any event we have $c_1(K_X) \cdot \Sigma_j \geq 0$, which implies $c_1(K_X) \cdot \Sigma_j = 0$ because $\sum_j c_1(K_X) \cdot \Sigma_j = 0$.

(2) Since $\text{Fix}(g) \subset \text{Fix}(g^2)$ and g^2 is an involution in G_0 , we see immediately that g has only isolated fixed points, with local representations of either type $(1, 1)$, $(3, 3)$, or type $(1, 3)$. We shall denote by s_+ , s_- the number of fixed points of g of type $(1, 3)$ and type $(1, 1)$ or $(3, 3)$ respectively. In order to determine s_+ , s_- , we first compute with the Lefschetz fixed point theorem and the G -signature theorem. To this end, it is useful to observe that for the induced action of the involution g^2 on $H^2(X; \mathbb{R})$, the 1-eigenspace has dimension 14 and the (-1) -eigenspace has dimension 8. With this understood, if we denote by t_{\pm} the dimension of the (± 1) -eigenspace of g in $H^2(X; \mathbb{R})$, then $t_+ + t_- = 14$. Now the Lefschetz fixed point theorem and the G -signature theorem

(cf. [11]) give rise to the following system of equations

$$\begin{cases} 2 + t_+ - (14 - t_+) &= s_+ + s_- \\ 4(6 - t_+) &= -16 + 2s_+ + (-2)s_-, \end{cases}$$

where we use the assumption $g \in G_0$ so that $b_2^+(X/g) = 3$, and we use the fact that the signature defect at a fixed point of type $(1, 3)$ and type $(1, 1)$ or $(3, 3)$ is $2, -2$ respectively, and the signature defect at a fixed point of g^2 is 0. The solutions for s_+, s_- (note that $s_+ + s_- \leq 8$) are $s_+ = 4$ and $s_- = 0, 2$ or 4 .

We proceed further by exploiting the fact that the action of g can also be lifted to the spin structure, and because g^2 is of even type, g is also of even type (i.e. a lifting of g to the spin structure is of order 4). Moreover, the induced lifting of g^2 to the spin structure is uniquely determined, i.e., it is independent of the different choices of liftings of g to the spin structure. With this understood, the computation of the “Spin-number” $Spin(g^2, X)$ plays a crucial role in the consideration which follows.

But first of all, a digression is needed in which we will recall a formula for the local contribution of a fixed point to the “Spin-number” (cf. Lemma 3.8 of [6]). Suppose h is an order p self-diffeomorphism ($p \geq 2$ and not necessarily prime) which is spin and almost complex. Then because of the h -equivariant almost complex structure, the h -equivariant spin structure corresponds to an h -equivariant complex line bundle L , such that at an isolated fixed point m of local representation type (a_m, b_m) , the weight r_m of the representation of h on the fiber of L at m obeys $2r_m + a_m + b_m \equiv 0 \pmod{p}$. Define $k(h, m) \equiv (2r_m + a_m + b_m)/p$. Then the local contribution of m to $Spin(h, X)$ is

$$I_m = (-1)^{k(h, m)+1} \cdot \frac{1}{4} \csc\left(\frac{a_m \pi}{p}\right) \csc\left(\frac{b_m \pi}{p}\right).$$

End of digression.

We apply the above formula to the involution $h \equiv g^2$. For each fixed point m of g^2 , $(a_m, b_m) = (1, 1)$, so that the local contribution $I_m = \pm \frac{1}{4}$, depending on whether $r_m = 0$ or 1 . Now suppose the g^2 -index of the Dirac operator as a character is

$$\text{index}_{g^2} D = d_0 + d_1 \mathbb{C}_1$$

where \mathbb{C}_k is the 1-dimensional weight- k representation. Then both d_0 and d_1 are even integers because of the quaternion structure. Since there are only 8 fixed points and each contributes $I_m = \pm \frac{1}{4}$ to $Spin(g^2, X)$, it follows easily that $Spin(g^2, X) = d_0 - d_1$ only takes values of $-2, 0$, or 2 . One can further eliminate the possibility of $Spin(g^2, X) = 0$ by observing that $d_0 + d_1 = -\text{sign}(X)/8 = 2$ and that both d_0, d_1 are even. The crucial consequence of the fact that $Spin(g^2, X)$ equals either -2 or 2 is that the weight r_m of the representation of g^2 on the fiber of the complex line bundle L is independent of the fixed point m . This implies that for the element g , either $s_+ = 0$ or $s_- = 0$. Since $s_+ = 4$, s_- must be 0, and the lemma follows. \square

Next we analyze the action of an element of G_0 of order 3. To this end, we recall from [5] that the connected components of $\cup_i C_i$ may be divided into the following three types:

- (A) A single J -holomorphic curve of self-intersection 0 which is either an embedded torus, or a cusp sphere, or a nodal sphere.
- (B) A union of two embedded (-2) -spheres intersecting at a single point with tangency of order 2.
- (C) A union of embedded (-2) -spheres intersecting transversely.

A type (C) component may be conveniently represented by one of the graphs of type \tilde{A}_n , \tilde{D}_n , \tilde{E}_6 , \tilde{E}_7 or \tilde{E}_8 listed in Figure 1, where a vertex in a graph represents a (-2) -sphere and an edge connecting two vertices represents a transverse, positive intersection point of the two (-2) -spheres represented by the vertices.

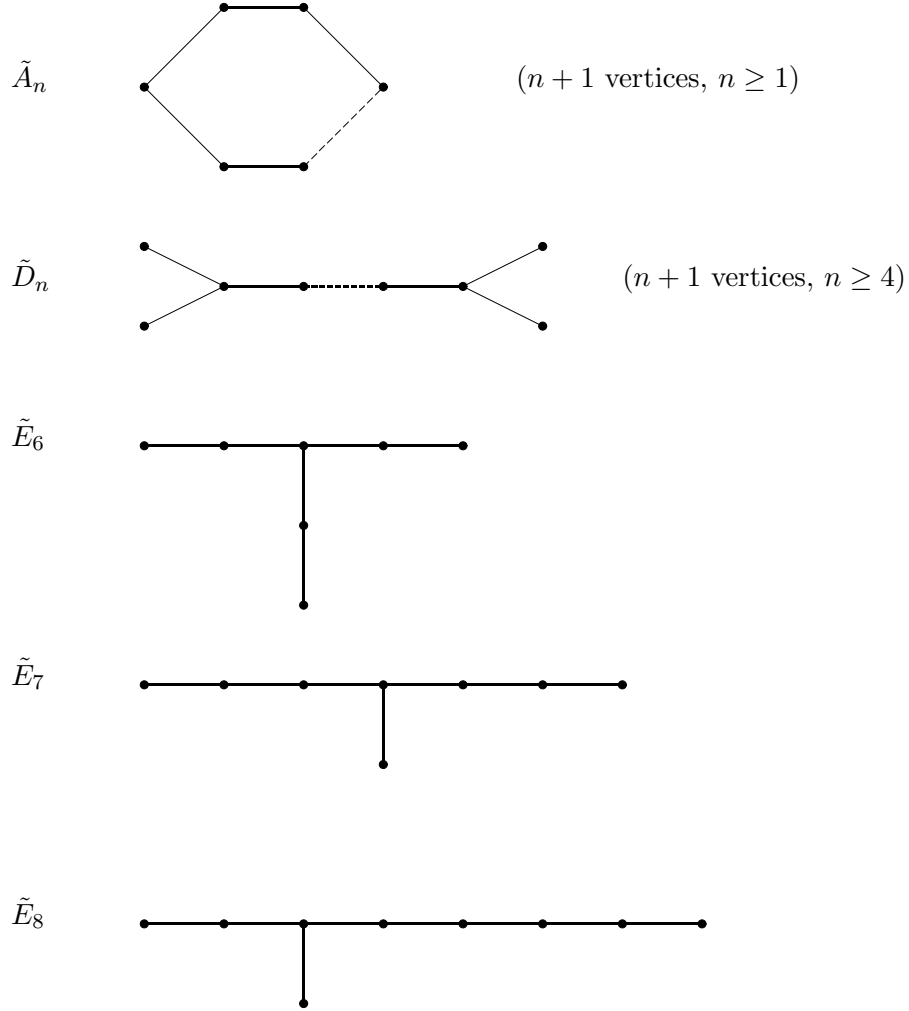


FIGURE 1.

Let $g \in G_0$ be an element of order 3. As we have demonstrated in [5], one can analyze the fixed point set structure of g by studying the induced action on $\cup_i C_i$. In particular, $\text{Fix}(g)$ may be divided into subsets (or groups) of the following four types.

- (I) One fixed point with local representation in $SL_2(\mathbb{C})$.
- (II) Three fixed points, all with local representation of type (k, k) for some $k \not\equiv 0 \pmod{3}$.
- (III) One fixed point of local representation type (k, k) , $k \not\equiv 0 \pmod{3}$, and one fixed spherical component of self-intersection -2 .
- (IV) One fixed toroidal component of self-intersection 0.

It is clear that a group of fixed points of type (III) comes only from a type (C) component of $\cup_i C_i$. For the sake of later arguments in this section, we shall give below a brief analysis of the action of g on a type (C) component of $\cup_i C_i$. Let Λ be a type (C) component which is invariant under g . We first note that there is an induced action of g on the graph representing Λ . Now suppose Λ is represented by a \tilde{A}_n graph. Then the induced action of g on the graph is either a trivial action or a rotation. In the former case, the fixed points of g contained in Λ are either entirely of type (I) or consist of $(n+1)/3$ groups of type (III) fixed points (cf. Lemma 3.6 and Prop. 3.7 in [5]). In the latter case, it is easily seen that either Λ contains no fixed points of g , or it is a union of three (-2) -spheres intersecting transversely at a single point, in which case the intersection point is the only fixed point of g contained in Λ and it is a type (I) fixed point. Suppose Λ is represented by a \tilde{D}_n graph. Then the induced action on the graph must be trivial, and the fixed points of g contained in Λ consist of 1 group of type (II) fixed points and $(n-1)/3$ groups of type (III) fixed points. If Λ is of type \tilde{E}_6 , then by a similar analysis as above we see that the induced action of g on the \tilde{E}_6 graph can not be trivial, and there are exactly two fixed points of g that are contained in Λ , which are on the (-2) -sphere represented by the central vertex. By Lemma 3.6 in [5] it follows easily that these two fixed points have local representations contained in $SL_2(\mathbb{C})$, hence they are of type (I). If Λ is represented by a \tilde{E}_7 graph, then the induced action on the graph must be trivial and Λ gives rise to 3 groups of type (III) fixed points of g . If Λ is represented by a \tilde{E}_8 graph, then Λ can not be invariant under g . See [5] for details. In summary, only a type \tilde{A}_n , \tilde{D}_n , or \tilde{E}_7 component of $\cup_i C_i$ can possibly contain a group of type (III) fixed points of g .

Lemma 2.3. *Suppose $g \in G_0$ is an element of order 3. Let u, v and w be the number of groups of type (I), (II) and (III) fixed points of g respectively, and let $t = b_2(X/g)$. Then*

- (1) $2u + 3v = 12$, $w \leq 6$ and $t \geq 10$. Moreover, $t = 10$ iff $(u, v, w) = (6, 0, 0)$.
- (2) Suppose $w = 0$. If there exist 3 distinct involutions $h_1, h_2, h_3 \in G_0$ each of which commutes with g , then $(u, v) = (6, 0)$.
- (3) Suppose $w = 0$. If g is contained in a subgroup of G_0 which is isomorphic to T_{24} , then $(u, v) = (6, 0)$.

Proof. (1) Note that a toroidal fixed component Y of g does not make any contribution in the Lefschetz fixed point theorem because $\chi(Y) = 0$, nor does it contribute in the

G -signature theorem because $Y \cdot Y = 0$. Hence we shall ignore it in our calculations below.

Observe that $t = b_2(X/g)$ is the dimension of the 1-eigenspace of g in $H^2(X; \mathbb{R})$, and that $t - (22 - t)/2$ is the trace of g on $H^2(X; \mathbb{R})$. Hence the Lefschetz fixed point theorem and the G -signature theorem give rise to the following equations

$$\begin{cases} 2 + t - (22 - t)/2 &= u + 3v + 3w \\ 3(6 - t) &= -16 + \frac{2}{3}u - 2v - 6w \end{cases}$$

where we make use of $b_2^+(X/g) = 3$ and that the total signature defect for a group of type (I), (II) and (III) fixed points is $\frac{2}{3}$, -2 and $\frac{3^2-1}{3} \cdot (-2) - \frac{2}{3} = -6$ respectively. The equation $2u + 3v = 12$ follows immediately, which has 3 solutions: $(u, v) = (6, 0), (3, 2)$, and $(0, 4)$. The inequality $w \leq 6$ follows from $u + 3v \geq 6$ and the fact that $t \leq b_2(X) = 22$. It is also easy to check that $t \geq 10$, with $t = 10$ iff $(u, v, w) = (6, 0, 0)$.

(2) Since each h_i has 8 isolated fixed points, and there is an induced action of g on $\text{Fix}(h_i)$, g and h_i must have at least 2 common fixed points. Now suppose that $(u, v) = (0, 4)$ and hence g has 12 isolated fixed points. From [5] and the assumption $w = 0$ we know that these 12 points are contained in 4 toroidal components of $\cup_i C_i$, each of which contains 3 isolated fixed points of g . Since g and h_i have common fixed points, there is at least one such toroidal component which is invariant under h_i . Consequently g and h_i generate an effective cyclic action of order 6 on that torus, which is known to have only 1 fixed point. This implies that the two distinct common fixed points of g and h_i are contained in two different toroidal components of $\cup_i C_i$. It follows easily that there are i, j with $i \neq j$ such that h_i and h_j leave one of the toroidal components invariant, because for each i , g and h_i have at least 2 common fixed points and there are totally 4 toroidal components of $\cup_i C_i$ containing the fixed points of g . But this is easily seen a contradiction, as h_i acts freely on the set of common fixed points of g and h_j because $\text{Fix}(h_i) \cap \text{Fix}(h_j) = \emptyset$. The case where $(u, v) = (3, 2)$ can be similarly eliminated. This proves that $(u, v) = (6, 0)$.

(3) Note that $T_{24} = Q_8 \rtimes_{\phi} \mathbb{Z}_3$, where we may assume without loss of generality that the action of $\mathbb{Z}_3 = \langle g \rangle$ on

$$Q_8 = \{i, j, k | i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j\}$$

is given by $\phi(g)(i) = j$, $\phi(g)(j) = k$ and $\phi(g)(k) = i$.

By Lemma 2.2, it follows easily that Q_8 has either 2 or 4 isolated fixed points (see e.g. [26]). Since there is an induced action of g on the fixed point set of Q_8 , we see immediately that T_{24} has at least 1 fixed point.

Now suppose $(u, v) = (0, 4)$. As we argued in (2) above, at least one of the 4 toroidal components must be invariant under T_{24} because it contains a fixed point of T_{24} . But this is impossible as there is no such a T_{24} -action on the torus (cf. [23]).

If $(u, v) = (3, 2)$, then g and $-1 \in Q_8$ must have 5 common fixed points. It follows as we argued in (2) above that each of the 2 toroidal components of $\cup_i C_i$ which contains the type (II) fixed points of g must be invariant under -1 , each containing exactly 1 common fixed point of g and -1 . But on the other hand, by the analysis in [5], each of the 2 toroidal components contains exactly 4 fixed points of -1 , so that all of the fixed points of -1 are contained in there. This is a contradiction to the fact that the

3 type (I) fixed points of g , which are not contained in the 2 toroidal components, are also fixed under -1 . The case where $(u, v) = (3, 2)$ is also ruled out. \square

Proof of Main Theorem:

Case (1). $G = L_2(7)$. First note that $G_0 = G$, i.e., $b_2^+(X/G) = 3$.

Lemma 2.4. *Let $g \in G = L_2(7)$ be any element of order 7. Then g has exactly 3 isolated fixed points, and is either pseudofree or has at most toroidal fixed components.*

Proof. We first show that there are no type (C) components of $\cup_i C_i$ which contain a fixed point of g such that its local representation is not in $SL_2(\mathbb{C})$. To this end, we recall that the normalizer of $\langle g \rangle$ in G is a maximal subgroup D of order 21 which is a semi-direct product of \mathbb{Z}_7 by \mathbb{Z}_3 (cf. [7]). Now let Λ be such a type (C) component. If it is represented by a type $\tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ or \tilde{E}_8 graph, then since the (-2) -spheres in Λ generate a lattice in $H_2(X; \mathbb{Z})$ which contains a negative-definite sublattice of rank at least 4, the orbit of Λ under the action of G can have at most 4 components because of the constraint $b_2^-(X) = 19$. On the other hand, one can easily check that Λ is not invariant under $G = L_2(7)$, and since the index of the maximal subgroup D is 8, there are at least 8 components in the orbit of Λ , which is a contradiction. If Λ is represented by a type \tilde{A}_n graph, then $n \equiv -1 \pmod{7}$, so that Λ contains at least 7 (-2) -spheres (cf. [5]). This is also not allowed by a similar argument as above. There is one more possibility that Λ is a union of three (-2) -spheres intersecting transversely at one single point. Note that the maximal subgroup D can not act linearly and freely on \mathbb{S}^3 , so that such a Λ can not be invariant under the action of D . Hence if such a Λ exists, there must be at least $3 \times 8 = 24$ components in the orbit of Λ under the action of G . But this is also impossible because of the constraint $b_2^-(X) = 19$. Hence there are no such type (C) components of $\cup_i C_i$.

Secondly, we claim that there are no type (B) components which contain a fixed point of g . If there is such a type (B) component, then since it also can not be invariant under the maximal subgroup D , there are at least 24 type (B) components in $\cup_i C_i$, which contradicts the fact that $b_2^-(X) = 19$. Hence there are no type (B) components containing a fixed point of g either.

With the above understood, the analysis in [5] shows that g has at most fixed toroidal components, and that the isolated fixed points of g are divided into groups of the following two types:

- (1) One fixed point with local representation in $SL_2(\mathbb{C})$.
- (2) Two fixed points with local representation of type $(2k, 3k), (-k, 6k)$ for some $k \not\equiv 0 \pmod{7}$ respectively.

Denote by t the dimension of the 1-eigenspace of g in $H^2(X; \mathbb{R})$ (note that $22 - t$ must be divisible by 6), and denote by u, v the number of groups of type (1), (2) isolated fixed points of g respectively. Then by the Lefschetz fixed point theorem and the G -signature theorem,

$$\begin{cases} 2 + t - (22 - t)/6 &= u + 2v \\ 7(6 - t) &= -16 + 10u - 8v, \end{cases}$$

where we use the fact that $b_2^+(X/g) = 3$ and that the total signature defect for a group of type (1), (2) fixed points is 10 and -8 respectively (cf. [5], Lemma 3.8). The solutions of the above system are

$$(t, u, v) = (4, 3, 0), (10, 2, 4), (16, 1, 8), (22, 0, 12).$$

The cases where $(t, u, v) = (10, 2, 4)$ or $(16, 1, 8)$ can be ruled out as follows. The maximal subgroup D induces a \mathbb{Z}_3 -action on the set of isolated fixed points of g , which must be free because D can not act freely and linearly on \mathbb{S}^3 . This implies that the number of fixed points, which is $u + 2v$, must be divisible by 3.

The case $(t, u, v) = (22, 0, 12)$ means that g is homologically trivial. Since $G = L_2(7)$ is a simple group, this implies that the action of G is also homologically trivial. But this is impossible by McCooley's theorem [18] because G is nonabelian.

The only case left is $(t, u, v) = (4, 3, 0)$, which shows that g has exactly 3 isolated fixed points. □

Next we consider the action of an element $g \in G$ of order 3. We claim that g has exactly 6 isolated fixed points, with possibly some fixed toroidal components. To see this, we note that there is an element $h \in G$ of order 7 such that $D = \langle g, h \rangle$ is a nonabelian subgroup of order 21, which is the normalizer of $\langle h \rangle$ (cf. [7]). From the proof of the above lemma, we see that the dimension of the $\exp(\frac{2\pi i k}{7})$ -eigenspace of h in $H^2(X; \mathbb{R})$ is $\frac{22-4}{6} = 3$ for each $1 \leq k \leq 6$. By examining the action of D on the $\exp(\frac{2\pi i k}{7})$ -eigenspaces of h , $1 \leq k \leq 6$, one can check easily that the dimension of the 1-eigenspace of g in $H^2(X; \mathbb{R})$ is at most 10. By Lemma 2.3 (1), our claim follows.

Now with Lemma 2.2, which describes the number of fixed points of an element of order 2 or 4, we see that for any $g \in G$, the Lefschetz fixed point theorem implies that the trace of g on $H^*(X; \mathbb{R})$, denoted by $tr(g)$, is the same as the trace of a symplectic automorphism of order $|g|$ on a $K3$ surface. This implies that

$$\dim(H^*(X; \mathbb{R}))^G = \mu(G) \equiv \frac{1}{|G|} \sum_{g \in G} tr(g) = 5.$$

(See [21] for the calculation of $\mu(G)$ for a symplectic automorphism group of a $K3$ surface.) This in turn implies that $H^2(X; \mathbb{R})^G$ is positive-definite because

$$\dim(H^2(X; \mathbb{R}))^G = \dim(H^*(X; \mathbb{R}))^G - 2 = 5 - 2 = 3,$$

and $b_2^+(X/G) = 3$. On the other hand, $c_1(K_X) \in H^2(X; \mathbb{R})^G$ and $c_1(K_X) \cdot c_1(K_X) = 0$, which implies that $c_1(K_X) = 0$ because $H^2(X; \mathbb{R})^G$ is positive-definite.

End of Case (1).

Case (2). $G = M_{20}$ or A_6 .

Lemma 2.5. *Suppose $H \subset G_0$ is a subgroup isomorphic to either A_5 or A_6 . Let $g \in H$ be an element of odd order. Then g is either pseudofree or has at most toroidal fixed components. Moreover, g has 4 isolated fixed points if $|g| = 5$, and g has either 6 or 12 isolated fixed points when $|g| = 3$.*

Proof. Let $g \in H$ be an element of order 5. Without loss of generality we may assume that $H \cong A_5$, because in the case of $H \cong A_6$, g is contained in an A_5 -subgroup of H . With this understood, the maximal subgroup of H containing g is a dihedral group $D_{10} \subset H$ of index 6 (cf. [7]). One can similarly argue, as in the proof of Lemma 2.4, that there are no type (C) components of $\cup_i C_i$ which contain a fixed point of g of local representation not in $SL_2(\mathbb{C})$.

By the analysis in [5], there are at most toroidal fixed components of g , and the isolated fixed points of g can be divided into groups of the following two types:

- (1) One fixed point with local representation in $SL_2(\mathbb{C})$.
- (2) Three fixed points, one with local representation of type $(k, 2k)$ and the other two of type $(-k, 4k)$ for some $k \not\equiv 0 \pmod{5}$.

Denote by t the dimension of the 1-eigenspace of g in $H^2(X; \mathbb{R})$ (note that $22 - t$ must be divisible by 4), and denote by u, v the number of groups of type (1), (2) isolated fixed points of g respectively. Then by the Lefschetz fixed point theorem and the G -signature theorem,

$$\begin{cases} 2 + t - (22 - t)/4 &= u + 3v \\ 5(6 - t) &= -16 + 4u - 8v, \end{cases}$$

where we use the fact that $b_2^+(X/g) = 3$ and that the total signature defect for a group of type (1), (2) fixed points is 4 and -8 respectively (cf. [5], Lemma 3.8). The solutions of the above system are

$$(t, u, v) = (6, 4, 0), (10, 3, 2), (14, 2, 4), (18, 1, 6), (22, 0, 8).$$

The cases where $u = 1$ or 3 can be eliminated as follows. There is an involution on the set of isolated fixed points of g induced by the action of D_{10} , which is free because D_{10} can not act freely and linearly on \mathbb{S}^3 , so that the number of isolated fixed points of g must be divisible by 2. To eliminate the case where $(t, u, v) = (14, 2, 4)$, note that in this case $\cup_i C_i$ has 4 type (B) components each of which contains a fixed point of g . But this is impossible because the index of $D_{10} \subset H$ is 6 and $b_2^-(X) = 19$. The case where $(t, u, v) = (22, 0, 8)$ is ruled out by McCooey's theorem [18] because H is simple and nonabelian. Hence g has 4 isolated fixed points when $|g| = 5$.

Next suppose $g \in H$ is an element of order 3, where H is either A_5 or A_6 . We claim that $\text{Fix}(g)$ does not contain any group of type (III) fixed points (i.e., $w = 0$ in Lemma 2.3). To see this, note first that $\cup_i C_i$ has no type (C) components represented by a graph of type $\tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ or \tilde{E}_8 . This is because if such a component does exist, there are at least 5 of them, as an order 5 element of H can not leave such a component invariant by the analysis above. But this contradicts $b_2^-(X) = 19$. Hence if $\text{Fix}(g)$ contains a group of type (III) fixed points, it must come from a type (C) component Λ which is represented by a type \tilde{A}_n graph, where $n \equiv -1 \pmod{3}$. Let $h \in H$ be an order 5 element. Since $g \neq hgh^{-1}$, it follows easily that h and g can not have a common isolated fixed point in Λ , which implies that either Λ is not invariant under h or h acts freely on Λ . In any event, the case $n > 2$ is ruled out similarly by the fact $b_2^-(X) = 19$. To eliminate the case where $n = 2$, we note that there is a subgroup $K \subset H$ which is isomorphic to the symmetric group S_3 and contains $\langle g \rangle$ as a normal subgroup. Clearly K can not leave Λ invariant if it is represented by a \tilde{A}_2 graph, so

that Λ must come in pairs. Again this is impossible by the fact that $b_2^-(X) = 19$. Hence $\text{Fix}(g)$ does not contain any group of type (III) fixed points. The action of K on $\text{Fix}(g)$ also implies that u is even in Lemma 2.3 (because S_3 can not act freely and linearly on \mathbb{S}^3). Hence g has either 6 or 12 isolated fixed points when $|g| = 3$.

Finally, note that g is either pseudofree or has at most toroidal fixed components. \square

Let $G = M_{20}$. Since $[G, G] = G$, we see that $G_0 = G$. We claim that for each $g \in G$ the trace $\text{tr}(g)$ on $H^*(X; \mathbb{R})$ is the same as that of a symplectic automorphism of order $|g|$ on a $K3$ surface. With this the proof of the main theorem proceeds identically as in the case of $L_2(7)$, as for $G = M_{20}$, $\mu(G) \equiv |G|^{-1} \sum_{g \in G} \text{tr}(g) = 5$ is also true for a holomorphic action (cf. [21]). When $|g| \neq 3$ or 6, our claim follows readily from Lemma 2.2 and Lemma 2.5. For the case where $|g| = 3$ or 6, we need to argue with some extra information about the structure of $G = M_{20}$.

According to Mukai [21], page 189, $M_{20} = 2^4 A_5$, where the action of A_5 on 2^4 is obtained by realizing 2^4 as the hypersurface $V = \{(a_i) \mid \sum_{i=1}^5 a_i = 0\} \subset (\mathbb{Z}_2)^5$ with A_5 acting as permutations of the 5 coordinates. Clearly, for each element g of order 3 in A_5 , there are 3 nonzero elements of V which are fixed under g . This gives 3 distinct involutions in G , each of which commutes with g . By Lemma 2.3 (2), g has 6 isolated fixed points. It also follows easily from the proof of Lemma 2.3 (2) that an order 6 element of G has 2 isolated fixed points, with possibly some fixed toroidal components. In conclusion, for an order 3 or 6 element $g \in G$, the trace $\text{tr}(g)$ on $H^*(X; \mathbb{R})$ is also the same as that of a symplectic automorphism on a $K3$ surface of the same order. This completes the proof for the case where $G = M_{20}$.

Let $G = A_6$. In this case, we also have $G_0 = G$. As above, it suffices to show that for each $g \in G$ with $|g| = 3$, there are 6 isolated fixed points. (Note that $\mu(G) \equiv |G|^{-1} \sum_{g \in G} \text{tr}(g) = 5$ is true for a holomorphic A_6 -action (cf. [21])). To this end, we recall the following fact about A_6 : There are 2 conjugacy classes of elements of order 3 in A_6 ; the centralizer of each order 3 element in A_6 is isomorphic to $(\mathbb{Z}_3)^2$, hence has order 9. Now suppose an element g of order 3 in $G = A_6$ is, instead, of 12 isolated fixed points. Then the conjugacy class of g will make an increase of $\frac{6}{9}$ to

$$\mu(G) \equiv \frac{1}{|G|} \sum_{g \in G} \text{tr}(g)$$

when compared with a holomorphic A_6 -action. Since there are only two conjugacy classes of elements of order 3 in A_6 , a nonzero increase to $\mu(G)$ is either $\frac{2}{3}$ or $\frac{4}{3}$, neither of which is integral. This shows that an element of order 3 in G must have 6 isolated fixed points, and the proof of the main theorem for the case of $G = A_6$ follows.

End of Case (2) where $G = M_{20}$ or A_6 .

Case (3). $G = A_{4,4}$. Let $H \equiv [G, G] = A_4 \times A_4$. Then since $[G, G] \subset G_0$, we have $b_2^+(X/H) = 3$. Note that

$$\mu(H) \equiv \frac{1}{|H|} \sum_{g \in H} \text{tr}(g) = 5$$

for a symplectic automorphism group H of a $K3$ surface (cf. [26]). Hence by Lemma 2.2, it suffices to show that for each $g \in H$ of order 3, the trace $tr(g)$ on $H^*(X; \mathbb{R})$ is the same as that of a symplectic automorphism of order 3 on a $K3$ surface.

There are 4 conjugacy classes of order 3 elements in G , which are represented by $(g, 1), (1, g), (g, g), (g, g^2) \in A_4 \times A_4 = H$ for some fixed element $g \in A_4$ of order 3. Since the trace on $H^*(X; \mathbb{R})$ only depends on the conjugacy class in G , it suffices to examine these 4 elements of H .

We first show that there are no type (III) fixed points (i.e., $w = 0$ in Lemma 2.3). Consider the case $(g, 1)$ first. The normalizer of $\langle (g, 1) \rangle$ in H is $\langle g \rangle \times A_4$ which has index 4. If Λ is a type (C) component of $\cup_i C_i$ which contains a group of type (III) fixed points of $(g, 1)$, then the fact $b_2^-(X) = 19$ immediately rules out the possibility that Λ is represented by a \tilde{E}_7 graph or a \tilde{A}_n graph where $n \neq 2$. If Λ is represented by a \tilde{D}_n graph or a \tilde{A}_2 graph, then one can check easily that the orbit of Λ under the normalizer $\langle g \rangle \times A_4$ has at least 3 components. This also contradicts $b_2^-(X) = 19$, and hence there are no type (III) fixed points of $(g, 1)$. The case of $(1, g)$ is completely parallel. For the case of (g, g) or (g, g^2) , the normalizer of $\langle (g, g) \rangle$ or $\langle (g, g^2) \rangle$ in H is $\langle g \rangle \times \langle g \rangle$ which has index 16. It follows immediately from $b_2^-(X) = 19$ that there are no type (III) fixed points.

Now by Lemma 2.3 (2), each of $(g, 1)$ and $(1, g)$ has exactly 6 isolated fixed points. The case of (g, g) or (g, g^2) is more involved, which is addressed in the following

Lemma 2.6. *Suppose $c_1(K_X) \neq 0$. Then the number of isolated fixed points of (g, g) or (g, g^2) is even.*

Proof. We consider the case of (g, g) only. The argument for (g, g^2) is completely parallel.

By Lemma 2.3, the number of isolated fixed points of (g, g) is either 6, 9 or 12. Suppose to the contrary that it is 9. A contradiction is derived as follows. Observe that there is an involution $h \in G \setminus H$ such that h and (g, g) generate a subgroup K of G , where K is isomorphic to S_3 and $\langle (g, g) \rangle$ is a normal subgroup of K . There is an induced action of K on $\text{Fix}((g, g))$, which preserves the type of the fixed points. Since (g, g) has 3 type (I) fixed points, one of them, denoted by p , must be fixed by K . Since $h \in G \setminus H$, $\text{Fix}(h)$ consists of a disjoint union of embedded J -holomorphic curves $\{\Sigma_j\}$ such that $c_1(K_X) \cdot \Sigma_j = 0$ for each j (cf. Lemma 2.2 (1)). It follows easily that there are fixed components $\Gamma_0, \Gamma_1, \Gamma_2$ of the three involutions h, ghg^{-1}, g^2hg^{-2} of K respectively, which intersect transversely at p and have the same genus and self-intersection. We claim that $\Gamma_0, \Gamma_1, \Gamma_2$ are (-2) -spheres, and consequently $(\sum_{k=0}^2 \Gamma_k)^2 = 0$. To see that each Γ_k is a (-2) -sphere, it suffices to show that $\Gamma_k^2 < 0$ because $c_1(K_X) \cdot \Gamma_k = 0$. Suppose to the contrary that $\Gamma_k^2 \geq 0$. Then $(\sum_{k=0}^2 \Gamma_k)^2 > 0$, which is not possible when $c_1(K_X) \neq 0$. To see this, note that all three classes $\sum_{k=0}^2 \Gamma_k$, $c_1(K_X)$, and the symplectic structure ω are fixed under K . Since $b_2^+(X/K) = 1$, we may write

$$\sum_{k=0}^2 \Gamma_k = a_1 \omega + \alpha_1, \quad c_1(K_X) = a_2 \omega + \alpha_2$$

for some $a_1, a_2 \in \mathbb{R}^+$ and $\alpha_1, \alpha_2 \in H^2(X; \mathbb{R})$ such that $\alpha_i \cdot \omega = 0$ and $\alpha_i^2 < 0$ for $i = 1, 2$. Without loss of generality we assume that $\omega^2 = 1$. Then $(\sum_{k=0}^2 \Gamma_k)^2 > 0$, $c_1(K_X)^2 = 0$, and $c_1(K_X) \cdot \sum_{k=0}^2 \Sigma_k = 0$ give rise to

$$a_1^2 + \alpha_1^2 > 0, \quad a_2^2 + \alpha_2^2 = 0, \quad \text{and} \quad a_1 a_2 + \alpha_1 \cdot \alpha_2 = 0.$$

We arrive at a contradiction to the triangle inequality

$$|\alpha_1 \cdot \alpha_2| = a_1 a_2 > (\alpha_1^2 \cdot \alpha_2^2)^{1/2}.$$

Hence $\Gamma_0, \Gamma_1, \Gamma_2$ are (-2) -spheres, and consequently $(\sum_{k=0}^2 \Gamma_k)^2 = 0$.

We claim that $\sum_{k=0}^2 \Gamma_k = \lambda c_1(K_X)$ for some $\lambda > 0$. To see this, let H^+ be the space of self-dual harmonic 2-forms. Then since $b_2^+(X/K) = 1$, the projections of the classes of $\sum_{k=0}^2 \Gamma_k$ and $c_1(K_X)$ into H^+ are linearly dependent. On the other hand, $\sum_{k=0}^2 \Gamma_k$ and $c_1(K_X)$ span an isotropic subspace because

$$\left(\sum_{k=0}^2 \Gamma_k\right)^2 = c_1(K_X)^2 = c_1(K_X) \cdot \sum_{k=0}^2 \Gamma_k = 0,$$

so that their projections into H^+ are injective. This proves the claim.

Now for each involution $h' \in H$, the set $h'(\cup_{k=0}^2 \Gamma_k)$ is disjoint from $\cup_{k=0}^2 \Gamma_k$ because of positivity of intersection of J -holomorphic curves and because

$$(h')^* \left(\sum_{k=0}^2 \Gamma_k\right) \cdot \left(\sum_{k=0}^2 \Gamma_k\right) = \lambda^2 (h')^* c_1(K_X) \cdot c_1(K_X) = \lambda^2 c_1(K_X)^2 = 0.$$

Since there are 15 distinct involutions in H , there must be 16 such configurations as $\cup_{k=0}^2 \Gamma_k$ which are mutually disjoint. This certainly contradicts $b_2^-(X) = 19$, and the lemma follows. \square

If $c_1(K_X) = 0$, then X is already “standard” and we are done in this case. Suppose $c_1(K_X) \neq 0$, then with the above lemma, we shall further argue that each of (g, g) or (g, g^2) must have 6 isolated fixed points. The reason is that if not, there will be an increase to $\mu(H) \equiv |H|^{-1} \sum_{g \in H} \text{tr}(g)$, in comparison with a symplectic automorphism group H of a $K3$ surface, of either $2 \times \frac{6}{9}$ or $4 \times \frac{6}{9}$, both of which are not integral. (The centralizer of (g, g) or (g, g^2) is $\langle g \times \times g \rangle$ which has order 9, and (g, g) , (g^2, g^2) , and (g, g^2) , (g^2, g) are not conjugate in H even though each pair of them are conjugate in G .) The proof for the case of $G = A_{4,4}$ is then completed.

End of Case (3) where $G = A_{4,4}$.

Case (4). $G = T_{192}$ or T_{48} . Set $H \equiv [G, G] \subset G_0$. Then in both cases,

$$\mu(H) \equiv |H|^{-1} \sum_{g \in H} \text{tr}(g) = 5$$

for a symplectic automorphism group H of a $K3$ surface (cf. [26]).

Let $G = T_{192}$. In this case $H = (Q_8 * Q_8) \times_{\phi} \mathbb{Z}_3$, where

$$Q_8 * Q_8 = Q_8 \times Q_8 / \langle (-1, -1) \rangle$$

is the central product of Q_8 with itself, and the action of \mathbb{Z}_3 on $Q_8 * Q_8$ is given by $\phi : x * y \mapsto \alpha^{-1}(x) * \alpha(y)$ for some fixed order 3 automorphism α of Q_8 (cf. [21]). The normalizer of \mathbb{Z}_3 in H is $\langle -1 \rangle \times \mathbb{Z}_3$, where $\langle -1 \rangle$ denotes the center of $Q_8 * Q_8$. It follows easily that for each $g \in \mathbb{Z}_3$, there are no type (III) fixed points of g because $b_2^-(X) = 19$ and the index of $\langle -1 \rangle \times \mathbb{Z}_3$ in H is 16. By Lemma 2.3 (3), each order 3 element of H has 6 isolated fixed points, with possibly some fixed toroidal components. Hence the case where $G = T_{192}$ follows.

Let $G = T_{48}$. Then H is isomorphic to $T_{24} = Q_8 \rtimes_{\phi} \mathbb{Z}_3$. By Lemma 2.3 (3), one only needs to verify that for any nontrivial element $g \in \mathbb{Z}_3$, there are no groups of type (III) fixed points of g .

Suppose to the contrary that there is a group of type (III) fixed points, which is contained in a type (C) component Λ . Observe that the normalizer of \mathbb{Z}_3 in H is $\langle -1 \rangle \times \mathbb{Z}_3$ which has index 4, it follows immediately from $b_2^-(X) = 19$ that Λ is not represented by a \tilde{E}_7 graph, or a \tilde{D}_n graph with $n > 4$, or a \tilde{A}_n graph with $n > 2$. In fact Λ is not represented by a \tilde{D}_4 graph either. The reason is that a \tilde{D}_4 type of Λ can not be invariant under $-1 \in Q_8$, because otherwise, the (-2) -sphere represented by the central vertex of Λ must contain 2 fixed points of -1 , which can not be fixed by any other element of Q_8 because Λ is not. But this contradicts the fact that by Lemma 2.2, Q_8 has at least 1 fixed point (cf. e.g. [26]), and -1 has only 8 isolated fixed points. This shows that the orbit of a \tilde{D}_4 type Λ under H has at least 8 components, contradicting $b_2^-(X) = 19$. Hence Λ is not represented by a \tilde{D}_4 graph.

It remains to eliminate the possibility that Λ is represented by a \tilde{A}_2 graph. Suppose this is the case. Then by the same argument as above, Λ can not be invariant under $-1 \in Q_8$, which means that Λ comes in pairs. Furthermore, the constraint $b_2^-(X) = 19$ allows for exactly two \tilde{A}_2 components, which give 2 groups of type (III) fixed points of g . To eliminate this possibility, we make use of the fact that there is an involution $h \in G \setminus H$, such that $hgh^{-1} = g^{-1}$. There is an induced action of h on the set of type (III) fixed points of g , where by replacing h with $(-1)h$, we may assume that h fixes the isolated fixed point in each of the 2 groups of type (III) fixed points. Since the local representation of g at the fixed point is of type (1,1) or (2,2), it follows that at the fixed point one has the commutativity $hg = gh$, which contradicts the fact that $hgh^{-1} = g^{-1}$. This finishes the proof that there are no groups of type (III) fixed points, and the case where $G = T_{48}$ follows.

3. PROOF OF THEOREM 1.1

In the proof of Theorem 1.1, the determination of the structure and the action of the subgroup G_0 follows the strategy of Xiao [26]. However, it relies on the fundamental work of Taubes [25] to establish the necessary properties of the action of G in order to implement Xiao's strategy.

The first half of Theorem 1.1 is contained in the following

Proposition 3.1. *Let X be a “standard” symplectic homotopy K3 surface, and let G be a finite group acting on X effectively and symplectically. Then there exists a short*

exact sequence of finite groups

$$1 \rightarrow G_0 \rightarrow G \rightarrow G^0 \rightarrow 1,$$

where G_0 is characterized as the maximal subgroup of G with property $b_2^+(X/G_0) = 3$ and G^0 is cyclic. Moreover, for each $g \in G_0$ the action of g is pseudofree with local representation at a fixed point contained in $SL_2(\mathbb{C})$, and the quotient orbifold X/G_0 can be smoothly resolved into a “standard” symplectic homotopy K3 surface.

Proof. Let ω be a symplectic structure on X which is preserved under G , and we fix an ω -compatible, G -equivariant almost complex structure J on X . Let K_X be the canonical bundle with the choice of J , and let g_J be the associated Riemannian metric, both of which are G -equivariant.

Following Taubes [25], we consider the following family (parametrized by $r > 0$) of perturbed Seiberg-Witten equations

$$D_A \psi = 0 \text{ and } P_+ F_A = \frac{1}{4} \tau(\psi \otimes \psi^*) + \mu,$$

where $\psi = \sqrt{r}(\alpha, \beta) \in \Gamma(K_X \oplus \mathbb{I})$, A is a $U(1)$ -connection on K_X , and

$$\mu = -\frac{ir}{4} \omega + P_+ F_{A_0}$$

for a canonical (up to gauge equivalence) connection A_0 on K_X^{-1} . According to [25], $c_1(K_X)$ is a Seiberg-Witten basic class, hence for any $r > 0$, there is a solution (ψ, A) with $\psi = \sqrt{r}(\alpha, \beta) \in \Gamma(K_X \oplus \mathbb{I})$. Moreover, as $r \rightarrow \infty$, the zero set $\alpha^{-1}(0) \subset X$ converges pointwise to a set of finitely many J -holomorphic curves with multiplicity, which represents the Poincaré dual of $c_1(K_X)$. Since X is “standard” by our assumption and hence $c_1(K_X) = 0$, we see that for sufficiently large $r > 0$, $\alpha^{-1}(0)$ must be empty. Consequently, by Proposition 4.4 in Taubes [25],

$$|q| \leq c \cdot r \cdot \exp(-c^{-1} \sqrt{r})$$

for some constant $c > 0$, with q standing for any of the quantities $r(1 - |\alpha|^2)$, $\sqrt{r} \nabla_a \alpha$ and F_a , where $a \equiv \frac{1}{2}(A - A_0)$ is a $U(1)$ -connection on K_X . Taking $r \rightarrow \infty$, α converges in C^∞ topology to a section $\alpha_0 \in \Gamma(K_X)$ with $|\alpha_0| = 1$, and the $U(1)$ -connection a converges to a flat connection a_0 on K_X , such that α_0 is parallel with respect to a_0 , i.e., $\nabla_{a_0} \alpha_0 = 0$. Finally, it is easily seen that (α_0, a_0) is unique up to gauge equivalence.

With the preceding understood, we note that since the family of perturbed Seiberg-Witten equations under consideration is G -equivariant (A_0 may be chosen such that $g^* A_0 = A_0$, $\forall g \in G$), the uniqueness of (α_0, a_0) up to gauge equivalence implies that for any $g \in G$, $g^* \alpha_0 = \phi(g) \alpha_0$ for some smooth circle-valued function $\phi(g) : X \rightarrow \mathbb{S}^1$. Since g is of a finite order, $\phi(g)$ must be a constant function because $\phi(g)^{|g|} = 1$. This gives rise to a homomorphism $\rho : G \rightarrow \mathbb{S}^1$ which is defined by $\rho : g \mapsto \phi(g) \in \mathbb{S}^1$. We define $G_0 \subset G$ to be the kernel of ρ and set $G^0 \equiv G/G_0$. Then clearly G^0 is cyclic. Moreover, if $g \in G$ has the property $b_2^+(X/g) = 3$, then as we argued in [5], the corresponding g -equivariant Seiberg-Witten invariant is nonzero, which implies $\phi(g) = 1$ and hence $g \in G_0$. Finally, we observe that for any $g \in G_0$, since α_0 is a nowhere vanishing section of K_X and $g^* \alpha_0 = \alpha_0$, g has at most isolated fixed points with a local representation contained in $SL_2(\mathbb{C})$.

It remains to show that the quotient orbifold X/G_0 can be smoothly resolved into a “standard” symplectic homotopy K3 surface. Note that this automatically implies $b_2^+(X/G_0) = 3$ as it equals the b_2^+ of the smooth resolution. In fact, in the next lemma we will prove an equivariant version of it, which finishes the proof of the proposition. \square

Consider a subgroup K of G which is contained in $G_0 = \ker \rho$ where $\rho : g \mapsto \phi(g)$, i.e., for any $g \in K$, $g^*\alpha_0 = \alpha_0$. Let H be a normal subgroup of K .

Lemma 3.2. *There exists a “standard” symplectic homotopy K3 surface X_H which is a smooth resolution of the orbifold X/H , such that K/H acts on X_H symplectically, extending the natural K/H -action on X/H under the resolution $X_H \rightarrow X/H$. Moreover, note that $b_2^+(X_H/(K/H)) = b_2^+(X/K) = b_2^+(X_K) = 3$.*

Proof. The construction of the smooth resolution of the symplectic orbifold X/H was given by McCarthy and Wolfson in [17]. We shall briefly review the procedure, indicating that it can be done equivariantly. In fact the construction is local, so we shall be focusing on a neighborhood of an isolated singular point of the orbifold, which by the equivariant Darboux’ theorem is modeled on \mathbb{C}^2/Γ , where Γ is the isotropy group at the singular point which acts complex linearly on \mathbb{C}^2 , and where the symplectic structure is given by the standard one on \mathbb{C}^2 , $\omega_0 = i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$.

Let U, V be the part of \mathbb{C}^2/Γ which lies outside and inside of the unit ball over Γ respectively, and let $W = \partial U = \partial V$ which is the 3-manifold \mathbb{S}^3/Γ . Since V is an algebraic surface with an isolated singularity, there is a nonsingular, minimal projective resolution $\pi : Y \rightarrow V$. Note that Y is Kähler. We let τ be a Kähler form on Y . Then for any $\epsilon > 0$, $\omega_\epsilon \equiv \pi^*\omega_0 + \epsilon\tau$ is a Kähler form on Y . We shall show that for a sufficiently small $\epsilon > 0$, the two pieces (U, ω_0) and (Y, ω_ϵ) can be symplectically “glued” together, which gives a smooth resolution of \mathbb{C}^2/Γ by a symplectic manifold.

To this end, we consider the contact structure ξ on W which is the distribution of complex lines in TW . Note that $\omega_0|_W = d\alpha$ for some contact form α such that $\xi = \ker \alpha$. On the other hand, since W is a rational homology 3-sphere, $\tau|_W = d\beta$ for a 1-form β , and hence $\omega_\epsilon|_W = d\alpha_\epsilon$ where $\alpha_\epsilon \equiv \alpha + \epsilon\beta$ is also a contact form when $\epsilon > 0$ is sufficiently small. By Moser’s argument, there exists a self-diffeomorphism $\psi : W \rightarrow W$ such that $\psi^*\alpha_\epsilon = e^f\alpha$ for some smooth function $f : W \rightarrow \mathbb{R}$. Pick a constant $C > 0$ such that $f < C$ on W . Let $Z \subset (W \times \mathbb{R}, d(e^t\alpha))$ be the symplectic “cylinder” defined by

$$Z \equiv \{(x, t) | x \in W, f(x) - C \leq t \leq 0\}.$$

Then the smooth resolution of \mathbb{C}^2/Γ by a symplectic manifold is given by

$$(X_{\epsilon, C}, \omega) \equiv (U, \omega_0) \cup (Z, d(e^t\alpha)) \cup (Y, e^{-C}\omega_\epsilon),$$

where the gluing between $\partial U = W$ and the component of ∂Z defined by $t = 0$ is by the identity map on W , and the gluing between the component of ∂Z defined by $t = f(x) - C$ and $\partial Y = W$ is by $(x, t) \mapsto \psi(x)$, where $\psi : W \rightarrow W$ is the self-diffeomorphism obtained above through Moser’s argument. We leave it to the reader to follow through that if a finite group Γ' acts complex linearly on \mathbb{C}^2/Γ , then there is a corresponding symplectic Γ' -action on the smooth resolution $(X_{\epsilon, C}, \omega)$.

(We remark that Moser's argument can be done equivariantly in the presence of a compact Lie group action; in particular, the self-diffeomorphism ψ of W can be made equivariant with respect to the Γ' -action on W , so that the gluing by $(x, t) \mapsto \psi(x)$ in the construction of $X_{\epsilon, C}$ is also equivariant.)

It remains to show that X_H is a “standard” symplectic homotopy $K3$ surface, and that $b_2^+(X_H/(K/H)) = 3$. The key step is the observation that X_H has a trivial canonical bundle. To see this, note that for any $g \in H$, since $g^*\alpha_0 = \alpha_0$, the nonzero section α_0 descends to a nonzero section $\hat{\alpha}_0$ of the canonical bundle of the symplectic orbifold X/H . With this it suffices to check it out locally, i.e., to show that the canonical bundle of $(X_{\epsilon, C}, \omega)$ is trivial.

On (U, ω_0) , the canonical bundle K_U is trivialized by $\hat{\alpha}_0$. On $(Z, d(e^t\alpha))$, the canonical bundle is the pull back of ξ^{-1} , the inverse line bundle of the contact structure ξ , via the projection $Z \rightarrow W$. Since $K_U|_W = \xi^{-1}$ and K_U is trivial, we see that K_Z is also trivial. Finally, K_Y is also trivial, because the symplectic form $e^{-C}\omega_\epsilon$ on Y is Kähler so that K_Y is simply given by the holomorphic canonical bundle. Since for each $g \in H$ the local representation at each fixed point of g is contained in $SL_2(\mathbb{C})$, the singularity of \mathbb{C}^2/Γ is a Du Val singularity, and it is known that in this case Y has a trivial canonical bundle if it is taken minimal. Now since $H^1(W; \mathbb{Z}) = 0$, a Mayer-Vietoris argument shows that the canonical bundle of $(X_{\epsilon, C}, \omega)$ is trivial.

As an immediate consequence, X_H is spin as $w_2(TX_H) = c_1(K_{X_H}) \pmod{2}$ must vanish. By Rohlin's theorem, $\text{sign}(X_H)$ is divisible by 16. Hence the intersection form on $H_2(X_H; \mathbb{Z})/T\text{or}$ is given by $m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2k(\pm E_8)$, with $m = b_2^+(X_H)$ and $k = |\text{sign}(X_H)|/16$. Now observe that X_H has at most a finite fundamental group, which implies that $b_1(X_H) = 0$. Hence

$$0 = c_1^2(K_{X_H}) = (2\chi + 3\text{sign})(X_H) = 2(2 + 2m + 16k) \pm 3 \cdot 16k = 0.$$

Since $m = b_2^+(X_H) = b_2^+(X/H) = 1$ or 3 (cf. Lemma 2.1), the only solution for (m, k) from the above equation is $m = 3$ and $k = 1$, and moreover, $\text{sign}(X_H) = -16$. This shows that X_H is a rational homology $K3$ surface. (Note that this conclusion also follows directly from Furuta's work on the $\frac{11}{8}$ -conjecture, cf. [9].)

Next we show that $\pi_1(X_H)$ is trivial. Let $\widehat{X_H}$ be the universal cover of X_H , which is compact because $\pi_1(X_H)$ is finite. Then $\widehat{X_H}$ is a closed, simply-connected symplectic 4-manifold with trivial canonical bundle. It is shown by Tian-Jun Li [15] that the Betti numbers of $\widehat{X_H}$ satisfy

$$b_2^+(\widehat{X_H}) \leq 3 \text{ and } b_2^-(\widehat{X_H}) \leq 19.$$

With $b_2^+(X_H) = 3$ and $b_2^-(X_H) = 19$ it follows easily that $\pi_1(X_H)$ is trivial. This completes the proof that X_H is a “standard” symplectic homotopy $K3$ surface.

Finally, we observe that

$$b_2^+(X_H/(K/H)) = b_2^+((X/H)/(K/H)) = b_2^+(X/K) = b_2^+(X_K) = 3.$$

□

Remark 3.3. The holomorphic version of Lemma 3.2 has been used in a fundamental way, first by Nikulin in [22] and then by Xiao in [26], to study finite symplectic

automorphism groups of $K3$ surfaces. In particular, following the argument in Nikulin [22], one can show, with Lemma 3.2, that for any $g \in G_0$, the order $|g| \leq 8$ and the number of fixed points of g is the same as that of an order $|g|$ symplectic automorphism of a $K3$ surface. However, we would like to point out that this statement can also be proved directly, by a lengthy argument involving essentially the various G -index theorems. Even though we have no need to pursue it here, but we would like to observe that $\text{Fix}(g) \neq \emptyset$ directly implies that the smooth resolution X_H in Lemma 3.2 is simply-connected, which is without appealing to Tian-Jun Li's result in [15] as we did in the proof of Lemma 3.2.

Now with Lemma 3.2 in place, we shall follow through the arguments of Xiao in [26] to complete the proof of Theorem 1.1 by showing that G_0 is a symplectic $K3$ group and that the action of G_0 on X has the same fixed point set structure as that of a corresponding symplectic automorphism group of a $K3$ surface.

In Section 1 of Xiao [26], the only argument involving complex geometry is in the proof of Lemma 2 there. We shall give a pure algebraic topology proof of this result below. In order to state the lemma, we first need to introduce the necessary notations.

Let X be a “standard” symplectic homotopy $K3$ surface and let G be a finite group acting effectively on X via symplectic symmetries such that $b_2^+(X/G) = 3$. Then as we have shown, X/G is a symplectic orbifold of only Du Val singularities, which has a smooth resolution X_G as defined in Lemma 3.2. Let L' be the sublattice of $H_2(X_G; \mathbb{Z})$ generated by the (-2) -spheres in X_G which are sent to the singular points under $X_G \rightarrow X/G$, and let L be the smallest primitive sublattice of $H_2(X_G; \mathbb{Z})$ containing L' . Then the analog of Lemma 2 in Xiao [26] is contained in the following lemma.

Lemma 3.4. $L/L' \cong G/[G, G]$.

Proof. Let A be a regular neighborhood of the (-2) -spheres in X_G which are mapped to the singular points under $X_G \rightarrow X/G$, and let $B = X_G \setminus A$ be the complement of A . Then the long exact sequence associated to the pair (X_G, A) gives rise to

$$H_2(A; \mathbb{Z}) \xrightarrow{i_*} H_2(X_G; \mathbb{Z}) \xrightarrow{j_*} H^2(B; \mathbb{Z}) \rightarrow 0,$$

where we have used the excision and Poincaré duality to make the identification $H_2(X_G, A; \mathbb{Z}) \cong H_2(B, \partial B; \mathbb{Z}) \cong H^2(B; \mathbb{Z})$, and we have used the fact that A is simply-connected so that $H_1(A; \mathbb{Z}) = 0$. On the other hand, by the universal-coefficient theorem for cohomology, we have the short exact sequence

$$0 \rightarrow \text{Ext}(H_1(B; \mathbb{Z}), \mathbb{Z}) \rightarrow H^2(B; \mathbb{Z}) \xrightarrow{h} \text{Hom}(H_2(B; \mathbb{Z}), \mathbb{Z}) \rightarrow 0.$$

Now observe that for any element $x \in H_2(X_G; \mathbb{Z})$, $h \circ j_*(x) = 0$ if and only if the intersection product of x with any element $y \in H_2(B; \mathbb{Z})$ is zero, which is precisely if and only if $x \in L$. This gives a surjective homomorphism $j_* : L \rightarrow \text{Ext}(H_1(B; \mathbb{Z}), \mathbb{Z})$ whose kernel is easily seen to be $L' = \text{Im}(i_*)$. Hence $L/L' \cong \text{Ext}(H_1(B; \mathbb{Z}), \mathbb{Z})$.

Finally, $\pi_1(B) = G$ so that $H_1(B; \mathbb{Z}) = G/[G, G]$ is a torsion group. This gives

$$L/L' \cong \text{Ext}(H_1(B; \mathbb{Z}), \mathbb{Z}) \cong H_1(B; \mathbb{Z}) = G/[G, G].$$

□

In Section 2 of Xiao [26], the author formulated a set of criteria obtained from Section 1, and by a computer search a list of possibilities for a symplectic $K3$ group as well as the combinatorial types of the actions were generated. A few of the cases were further eliminated to reach the final list, where the arguments are those in [26] which precedes Lemma 5. We observe that all these arguments can be used in the present situation without changing a word (of course with our Lemma 3.2 in place). This finishes the proof of Theorem 1.1.

Remark 3.5. The holomorphic version of Theorem 1.1 is contained in Nikulin [22]. There it was also shown that the order of the cyclic group G^0 is bounded by 66 (which is a sharp bound). The proof of this result involves arguments in complex geometry which are not available in the present, symplectic category. However, we should point out that there are further informations contained in the proof of Proposition 3.1 which can be used to analyze G^0 ; in particular, it is very likely that $|G^0|$ has a universal upper bound. We shall not pursue this issue here, but wish to point out that because of the homological rigidity of symplectic symmetries of a “standard” symplectic homotopy $K3$ surface established in [5], the prime factors in $|G^0|$ are bounded by $b_2 = 22$.

4. THE LATTICE L_X AND PROOF OF THEOREM 1.2

Recall that the Seiberg-Witten invariant of a simply-connected, closed, oriented, smooth 4-manifold M with $b_2^+ \geq 2$ is a map

$$SW_M : \{\beta \in H^2(M; \mathbb{Z}) \mid \beta \equiv w_2(TM) \pmod{2}\} \rightarrow \mathbb{Z}.$$

A class β is called a (Seiberg-Witten) basic class if $SW_M(\beta) \neq 0$. It is a fundamental fact that the set of basic classes is finite. Moreover, if β is a basic class, then so is $-\beta$ with

$$SW_M(-\beta) = (-1)^{(\chi + \text{sign})(M)/4} SW_M(\beta).$$

When M is symplectic, a fundamental result of Taubes says that the canonical class $c_1(K_X)$ associated to a symplectic structure is always a basic class. The Seiberg-Witten invariant SW_M is an invariant of the diffeomorphism class of M , whose sign depends on a choice of an orientation of

$$H^0(M; \mathbb{R}) \otimes \det H_+^2(M; \mathbb{R}).$$

In particular, the set of basic classes depends only on the diffeomorphism type of M . When M is a homotopy $K3$ surface, a theorem of Morgan and Szabó [20] says that $\beta = 0$ is always a basic class. Furthermore, when M is symplectic, work of Taubes [25] gives additional information about the Seiberg-Witten invariant, in particular, about the set of basic classes.

Let X be a symplectic homotopy $K3$ surface. We set

$$L_X \equiv \text{Span}(\beta \in H^2(X; \mathbb{Z}) \mid SW_X(\beta) \neq 0) \subset H^2(X; \mathbb{Z}),$$

and set $r_X \equiv \text{rank}(L_X)$. Let ω be any symplectic structure on X , and let K_X be the associated canonical bundle. Then Taubes [25] showed that $c_1(K_X) \in L_X$ and $0 \leq \beta \cdot [\omega] \leq c_1(K_X) \cdot [\omega]$ for any basic class β . In particular, $c_1(K_X) = 0$ iff $r_X = 0$.

Theorem 4.1. *Let X be a symplectic homotopy K3 surface. Then the lattice of basic classes L_X is isotropic, i.e., for any $x, y \in L_X$, the cup product of x and y is zero. As a consequence, the rank of L_X is bounded from above by $\min(b_2^+, b_2^-) = 3$, i.e., $r_X \leq 3$.*

Proof. Let ω be a symplectic structure of X , and let K_X be the canonical bundle. Since X is minimal, and $c_1^2(K_X) = 2\chi(X) + 3\text{sign}(X) = 0$, a theorem of Taubes (cf. [25], Theorem 0.2 (5)) says that for any basic class β , $e_\beta \equiv \frac{1}{2}(c_1(K_X) + \beta) \in H^2(X; \mathbb{Z})$ is Poincaré dual to $\sum_i m_i T_i$, where $m_i > 0$ and $\{T_i\}$ is a finite set of disjoint, symplectically embedded tori with self-intersection 0.

To see L_X is isotropic, it suffices to show that for any basic classes β, β' , the cup product $\beta \cdot \beta' = 0$, which follows from the generalized adjunction formula as follows. Suppose $e_\beta = \sum_i m_i T_i$ where $\{T_i\}$ is a finite set of disjoint, symplectically embedded tori with self-intersection 0. Then for any basic class β' , the generalized adjunction formula when applied to T_i asserts that

$$\text{genus}(T_i) \geq 1 + \frac{1}{2}(|\beta' \cdot T_i| + T_i^2).$$

This implies, for each i , $\beta' \cdot T_i = 0$ because $\text{genus}(T_i) = 1$ and $T_i^2 = 0$, and consequently, $e_\beta \cdot \beta' = 0$. In particular, since $c_1(K_X)$ is a basic class, we have $e_\beta \cdot c_1(K_X) = 0$, which implies that $\beta \cdot c_1(K_X) = 0$ for any basic class β . (This is because $e_\beta \equiv \frac{1}{2}(c_1(K_X) + \beta)$ and $c_1^2(K_X) = 0$.) Now we go back to $e_\beta \cdot \beta' = 0$, and conclude that

$$\beta \cdot \beta' = 2e_\beta \cdot \beta' - c_1(K_X) \cdot \beta' = 0.$$

Finally, we point out that $r_X \leq 3$ follows directly from the fact that the projection of L_X into $H_+^2(X; \mathbb{Z})$ is injective (because L_X is isotropic). □

Remark 4.2. Suppose G is a finite group which acts on a symplectic homotopy K3 surface X via symplectic symmetries. Then there is an induced action of G on the set of basic classes, which can be extended to a linear action on the lattice L_X . Moreover, let ω be the symplectic structure which is preserved under the action of G , and let K_X be the associated canonical bundle. Then $c_1(K_X) \in L_X$ is fixed under the action of G , and since ω is also fixed, the function $\omega : L_X \rightarrow \mathbb{R}$ defined by $x \mapsto [\omega] \cdot x$ is G -invariant. On the other hand, since the action of G on $H^0(X; \mathbb{R}) \otimes \det H_+^2(X; \mathbb{R})$ is orientation-preserving (cf. Lemma 2.1), one has, for any basic class β ,

$$SW_X(g \cdot \beta) = SW_X(\beta), \quad \forall g \in G.$$

It is clear that the induced action of G on L_X may be exploited to relate the action of G on X and the smooth structure of X .

Proof of Theorem 1.2

The construction of this type of exotic K3 surfaces is due to Fintushel and Stern, which is done by performing the knot surgery on three disjoint, homologically distinct, symplectically embedded tori in a Kummer surface (cf. [8], compare also [10]). Our observation here is that it can be done equivariantly. However, we would like to point out that the three tori (actually 12 tori divided into 3 groups) have to be chosen differently (cf. Remark 4.3).

Consider the 4-torus $T^4 = (\mathbb{S}^1)^4$ with the involution ρ , which is defined in the angular coordinates by

$$\rho : (\theta_0, \theta_1, \theta_2, \theta_3) \mapsto (-\theta_0, -\theta_1, -\theta_2, -\theta_3), \text{ where } \theta_j \in \mathbb{R}/2\pi\mathbb{Z}.$$

There are 16 isolated fixed points $(\theta_0, \theta_1, \theta_2, \theta_3)$ where each θ_j takes values in $\{0, \pi\}$. A Kummer surface is a smooth 4-manifold which is obtained by replacing each of the singular points in the quotient T^4/ρ with an embedded (-2) -sphere. We denote the 4-manifold by X_0 .

We shall give a more concrete description of X_0 below, where X_0 is also naturally endowed with a symplectic structure. Consider the symplectic form Ω on T^4 , which is equivariant with respect to the involution ρ :

$$\Omega \equiv \sum_{(i,j,k)} (d\theta_0 \wedge d\theta_i + d\theta_j \wedge d\theta_k)$$

where the sum is taken over $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. This gives rise to a symplectic structure on the orbifold T^4/ρ . One can further symplectically resolve the orbifold singularities to obtain a symplectic structure on X_0 as follows. By the equivariant Darboux' theorem, the symplectic structure is standard near each orbifold singularity. In particular, it is modeled on a neighborhood of the origin in $\mathbb{C}^2/\{\pm 1\}$ and admits a Hamiltonian \mathbb{S}^1 -action with moment map $\mu : (w_1, w_2) \mapsto \frac{1}{4}(|w_1|^2 + |w_2|^2)$, where w_1, w_2 are the standard coordinates on \mathbb{C}^2 . Fix a sufficiently small $r > 0$ and remove $\mu^{-1}([0, r))$ from T^4/ρ at each of its singular point. Then X_0 is diffeomorphic to the 4-manifold obtained by collapsing each orbit of the Hamiltonian \mathbb{S}^1 -action on the boundaries $\mu^{-1}(r)$, which is naturally a symplectic 4-manifold (cf. [14]). We denote the symplectic structure on X_0 by ω_0 .

Let $G = (\mathbb{Z}_2)^3$. We shall next describe a G -action on X_0 which preserves the symplectic structure ω_0 . Consider first the following G -action on T^4 :

$$a \cdot (\theta_0, \theta_1, \theta_2, \theta_3) = (\theta_0, \theta_1 + \pi a_1, \theta_2 + \pi a_2, \theta_3 + \pi a_3)$$

where $a = (a_1, a_2, a_3) \in G$ with each $a_j \in \mathbb{Z}_2 \equiv \mathbb{Z}/2\mathbb{Z}$. One can check easily that the above G -action commutes with the involution ρ , so that there is an induced G -action on the orbifold T^4/ρ . Moreover, the G -action clearly preserves the symplectic form

$$\Omega \equiv \sum_{(i,j,k)} (d\theta_0 \wedge d\theta_i + d\theta_j \wedge d\theta_k)$$

on T^4 , hence descends to a symplectic G -action on T^4/ρ . From the description of (X_0, ω_0) given in the previous paragraph, it follows easily that there is an induced, symplectic G -action on (X_0, ω_0) . (The key point here is that Lerman's symplectic cutting can be done equivariantly, cf. [14].) The G -action on X_0 is pseudofree; in fact, for any $0 \neq a = (a_1, a_2, a_3) \in G$, a fixed point of a in X_0 has angular coordinates $(\theta_0, \theta_1, \theta_2, \theta_3)$, where $\theta_0 = 0$ or π , and for $j = 1, 2, 3$, $\theta_j = 0$ or π if $a_j = 0$ and $\theta_j = \pi/2$ or $3\pi/2$ if $a_j = 1$. (Note that each $a \neq 0$ in G has 8 isolated fixed points.)

We shall next describe a set of 12 disjoint, symplectically embedded tori in (X_0, ω_0) , which is invariant under the G -action. The 12 tori are divided into 3 groups, labeled naturally by $j = 1, 2, 3$, and each group consists of 4 tori on which G acts freely and

transitively. For simplicity we shall only describe the group of tori indexed by $j = 1$ in detail; the others are completely parallel.

Consider the projection π_1 from T^4 to T^2 where

$$\pi_1 : (\theta_0, \theta_1, \theta_2, \theta_3) \mapsto (\theta_2, \theta_3).$$

For any fixed $\delta_{12}, \delta_{13} \in \mathbb{R}/2\pi\mathbb{Z}$ other than $0, \pi/2, 3\pi/2$ and π , the 4 tori in T^4

$$\begin{aligned} T_{1,0} &\equiv \pi_1^{-1}(\delta_{12}, \delta_{13}) & T_{1,1} &\equiv \pi_1^{-1}(\delta_{12} + \pi, \delta_{13}) \\ T_{1,2} &\equiv \pi_1^{-1}(\delta_{12}, \delta_{13} + \pi) & T_{1,3} &\equiv \pi_1^{-1}(\delta_{12} + \pi, \delta_{13} + \pi) \end{aligned}$$

are symplectically embedded with respect to the symplectic form

$$\Omega \equiv \sum_{(i,j,k)} (d\theta_0 \wedge d\theta_i + d\theta_j \wedge d\theta_k).$$

Moreover, they descent to 4 disjoint tori in T^4/ρ , and if the distance between δ_{12}, δ_{13} to $0, \pi/2, 3\pi/2$ and π is sufficiently large, $T_{1,k}$, $0 \leq k \leq 3$, can be regarded as tori in X_0 , which are disjoint and symplectically embedded. The union $\cup_k T_{1,k}$ is easily seen to be invariant under the action of G on X_0 . Moreover, the action of G on $\cup_k T_{1,k}$ is transitive, and each $T_{1,k}$ is invariant under an involution of G , which acts on the torus freely via translations.

In the same vein, one can consider projections

$$\pi_2 : (\theta_0, \theta_1, \theta_2, \theta_3) \mapsto (\theta_1, \theta_3) \text{ and } \pi_3 : (\theta_0, \theta_1, \theta_2, \theta_3) \mapsto (\theta_1, \theta_2)$$

and choose $\delta_{21}, \delta_{23}, \delta_{31}, \delta_{32} \in \mathbb{R}/2\pi\mathbb{Z} \setminus \{0, \pi/2, \pi, 3\pi/2\}$ to obtain 8 other tori $T_{j,k}$, where $j = 2, 3$ and $0 \leq k \leq 3$. One can check easily that under further conditions that

$$\delta_{13} \neq \pm\delta_{23}, \pm(\delta_{23} + \pi), \delta_{12} \neq \pm\delta_{32}, \pm(\delta_{32} + \pi), \delta_{21} \neq \pm\delta_{31}, \pm(\delta_{31} + \pi),$$

The 12 tori $T_{j,k}$ in X_0 are disjoint.

The exotic $K3$ surfaces are constructed by performing the Fintushel-Stern knot surgery on each of the 12 tori $T_{j,k}$ in X_0 with a fibered knot. The key issue here is that the knot surgery needs to be performed equivariantly with respect to the G -action on X_0 . To this end, we shall first give a brief review of the knot surgery from [8].

Let M be a simply-connected smooth 4-manifold with $b_2^+ > 1$, and let T be a c-embedded torus in M , i.e., T is a smooth fiber in a cusp neighborhood in X , which carries a nontrivial homology class in M . Consider a knot K in \mathbb{S}^3 , and let m denote a meridional circle to K . Let Y_K be the 3-manifold obtained by performing 0-framed surgery on K . Then m can also be viewed as a circle in Y_K . In $Y_K \times \mathbb{S}^1$ we have the smoothly embedded torus $T_m \equiv m \times \mathbb{S}^1$ of self-intersection 0. Since a neighborhood of m has a canonical framing in Y_K , a neighborhood of the torus T_m in $Y_K \times \mathbb{S}^1$ has a canonical identification with $T_m \times D^2$. With this understood, the knot surgery on T with knot K is the smooth 4-manifold M_K , which is the fiber sum

$$M_K \equiv M \#_{T=T_m} (Y_K \times \mathbb{S}^1) = [M \setminus (T \times D^2)] \cup [(Y_K \times \mathbb{S}^1) \setminus (T_m \times D^2)].$$

Here $T \times D^2$ is a tubular neighborhood of the torus T in M . The two pieces are glued together so as to preserve the homology class $[\text{pt} \times \partial D^2]$. Note that the diffeomorphism type of the fiber sum is not uniquely determined in general, and the 4-manifold M_K is taken to be any manifold constructed in this fashion. A basic theorem of Fintushel

and Stern states that M_K is naturally homeomorphic to M and the Seiberg-Witten invariants of the two manifolds are related by $sw_{M_K} = sw_M \cdot \Delta_K(t)$, where sw_{M_K} , sw_M are certain Laurent polynomials defined from the Seiberg-Witten invariants of M_K and M respectively, and $\Delta_K(t)$ is the Alexander polynomial of K , with $t = \exp(2[T])$. See [8] for more details. We remark that when M is symplectic and T is symplectically embedded, M_K can be naturally made symplectic by choosing any fibered knot K . Note that when M is the standard $K3$ surface, one has $sw_M = 1$, so that M_K is an exotic $K3$ surface as long as the knot K has a nontrivial Alexander polynomial.

With the preceding understood, note that in our present situation, each of the 12 tori $T_{j,k}$ is invariant under an involution of G . Moreover, the action on the tubular neighborhood $T_{j,k} \times D^2$ projects to a trivial action on the D^2 -factor. In order to do the knot surgery equivariantly, we shall consider the involution on $Y_K \times \mathbb{S}^1$ which is trivial on the Y_K -factor and is by translation on the \mathbb{S}^1 -factor. Recall that the only requirement in the knot surgery is to preserve the homology class $[\text{pt} \times \partial D^2]$ under the gluing. Since on the $Y_K \times \mathbb{S}^1$ side $\text{pt} \times \partial D^2$ is given by a 0-framed copy of the knot K in Y_K and the involution on $Y_K \times \mathbb{S}^1$ is chosen to be trivial on the Y_K -factor, it follows easily that for any fixed fibered knot K , one can do the knot surgery simultaneously on each of the 12 tori $T_{j,k}$ with the knot K , such that the G -action on X_0 can be extended to a symplectic G -action on the resulting 4-manifold X_K , which is a symplectic homotopy $K3$ surface with Seiberg-Witten invariant

$$sw_{X_K} = \Delta_K(t_1)^4 \Delta_K(t_2)^4 \Delta_K(t_3)^4$$

where $t_j = \exp(2[T_{j,0}])$. Note that the three tori $T_{1,0}$, $T_{2,0}$ and $T_{3,0}$ are homologically linearly independent, so that X_K has the maximal exoticness (i.e. $r_X = 3$). Moreover, by the nature of construction, the G -action on X_K is clearly pseudofree and induces a trivial action on the lattice L_X of the Seiberg-Witten basic classes.

Finally, we point out that one can choose an infinite family of distinct knots K such that the 4-manifolds X_K are distinct (for example, one may choose K with distinct genus).

Remark 4.3. We would like to explain why the tori in our construction have to be chosen differently from that in [8] or [10], and point out that for the same reason our construction can not be extended to the symplectic $K3$ group $(\mathbb{Z}_2)^4$. The key point here is that one has to make sure that each $T_{j,k}$ can only be invariant under a cyclic subgroup of G . Otherwise, we will be forced to introduce a nontrivial cyclic action on the factor Y_K in $Y_K \times \mathbb{S}^1$. Of course, one way to obtain such a cyclic action on Y_K is to pick a cyclic action on \mathbb{S}^3 under which K is invariant, and then do the 0-framed surgery on K equivariantly. The problem is that the action on the tubular neighborhood $T_{j,k} \times D^2$ projects to a trivial action on the D^2 -factor, and since under the knot surgery $\text{pt} \times \partial D^2$ is glued to a 0-framed copy of K in Y_K , the action on \mathbb{S}^3 which we picked at the beginning has to fix the knot K . However, by the Smith conjecture [19] this is not possible unless K is the unknot. With this understood, we remark that with the choice of tori as in [8] or [10], one can only construct a $(\mathbb{Z}_2)^2$ -action on a homotopy $K3$ surface with maximal exoticness. On the other hand, for the group $G = (\mathbb{Z}_2)^4$, our construction would not even yield an effective G -action on a homotopy $K3$ surface with nontrivial exoticness (i.e. $r_X > 0$).

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